Engineering Notes

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Inviscid Parallel Flow Stability with Mean Profile Distortion

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Introduction

LINEAR stability theory, as described by Lin,[1] predicts the growth of infinitesimal disturbances on prescribed parallel mean flows. However, over large time scales, the cumulative effects of wave-induced distortions to the mean flow and of higher order harmonics become important and a nonlinear theory may be required. For viscous flows a nonlinear theory assuming "near-equilibrium" changes to the mean profile is available.[2-4] A governing mean flow equation ignores the unsteady term to leading order, so that the momentum balance is taken, primarily, between the viscous term and the Reynolds stress. In this Note the inviscid problem is considered. Because viscosity is absent, the unsteady convective term previously discarded must now be retained to balance the Reynolds stress: "near-equilibrium" changes do not apply because the flow is inherently unsteady, thus resulting in a mathematical formulation not previously discussed. For simplicity, however, we will neglect all higher harmonics and consider only the effects of distortion to the mean flow as would be induced by the primary disturbance. The physical limitations implicit in this approximate model will be fully discussed later. Here, instead, we emphasize that the model is mathematically tractable, yielding closed-form results for both mean and disturbance flows that appear to be physically acceptable.

Analysis

Let x, y, and t be streamwise, transverse, and time variables, and $u=\Psi_y$ and $v=-\Psi_x$ be flow speeds derived from the normal mode streamfunction expansion

$$\Psi(x,y,t)=\varphi_0(y,t)+\sum_{j=1}^{\infty}\Big\{\varphi_j(y,t)e^{ijk(x-c_r t)} + \tilde{\varphi}_j(y,t)e^{-ijk(x-c_r t)}\Big\} \tag{1}$$

Here, k and c_r are real, while tildes indicate complex conjugates. Now substitute Eq. (1) in the inviscid vorticity equation and equate coefficients of like harmonics. This leads to a sequence of nonlinearly coupled equations, e.g.,

$$\left(\bar{U}-c_r-\frac{i}{k}\frac{\partial}{\partial t}\right)\left(\varphi_1''-k^2\varphi_1\right)-\bar{U}''\varphi_1+\ldots=0 \tag{2}$$

$$\frac{\partial^2\bar{U}}{\partial t\partial y}=\frac{\partial^2\tau}{\partial y^2}=-ik\frac{\partial^2}{\partial y^2}\left\{\left(\varphi_1'\tilde{\varphi}_1-\tilde{\varphi}_1'\varphi_1\right)+\ldots\right\} \tag{3}$$

where primes denote y derivatives, the "+..." refers to higher harmonic terms not shown, $\bar{U}(y,t)=\partial\varphi_0/\partial y$ is an unknown parallel flow which equals $U_l(y)$ in the absence of waves, and τ is the Reynolds stress. Assuming also that $\varphi_1(y,t)$ vanishes at $y=y_1$ and y_2, we seek the timewise behavior of the complete flow, given $U_l(y)$ and k.

Note that our $\varphi_1(y,t)$ can be related to the eigenfunction $\varphi(y)$ defined by Rayleigh's linear equation. If $c=c_r+ic_i$, where $\omega_i=kc_i$ is the linear growth rate, we have $\varphi_1(y,t)\cong\frac{1}{2}\varphi(y)\exp(kc_i t)$. We first solve Eq. (3) approximately by evaluating the right side with linear results, in particular, using

$$\tau=\tfrac{1}{2}kc_i e^{2kc_i t}\int_{y_1}^{y}\frac{U_l''|\varphi|^2}{|U_l-c|^2}\,dy \tag{4}$$

Next introduce an amplitude B such that $|\varphi|=|B|\,|\varphi_N|$ and

$$\int_{y_1}^{y_2}\left\{\left|\varphi_N'\right|^2+k^2\left|\varphi_N\right|^2\right\}dy=1$$

For example, the average wave energy density now satisfies

$$E=\tfrac{1}{4}e^{2kc_i t}\int_{y_1}^{y_2}\left\{\left|\varphi'\right|^2+k^2\left|\varphi\right|^2\right\}dy=\tfrac{1}{4}e^{2kc_i t}\left|B\right|^2$$

The expansion $\bar{U}(y,t)=U_l(y)+E(t)S(y)$ then leads to

$$\frac{dE}{dt}\frac{dS}{dy}=2\omega_i E\frac{\partial}{\partial y}\frac{U_l''\,|\varphi_N|^2}{|U_l-c|^2} \tag{5}$$

where $S(y)$ is a profile correction to be determined. Division by $dE/dt\cong 2kc_iE$ and the use of $S(y_1)=S(y_2)=0$ now shows that $S(y)=U_l''|\varphi_N|^2/|U_l-c|^2$. Thus, the distorted mean flow takes the form

$$\bar{U}(y,t)=U_l(y)+E\frac{U_l''\,|\varphi_N|^2}{|U_l-c|^2} \tag{6}$$

Thus, nonlinearity introduces a local velocity defect where $U_l''(y)<0$ and an excess where $U_l''(y)>0$. For inflectional instability with $c_i>0$, the Reynolds stress retards the flow above the inflection point and speeds it up below; it tends to smooth out the inflectional region and, as will be shown, stabilizes the primary disturbance. Throughout this process the momentum of the mean flow is constant. Equation (6) shows that

$$\int_{y_1}^{y_2}\bar{U}(y,t)\,dy=\int_{y_1}^{y_2}U_l\,dy+E\int_{y_1}^{y_2}\frac{U_l''\,|\varphi_N|^2}{|U_l-c|^2}\,dy$$

$$=\int_{y_1}^{y_2}U_l(y)\,dy=\text{constant} \tag{7}$$

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because the shear stress given in Eq. (4) vanishes at $y=y_2$.

The detailed wave energy growth can be obtained from the inviscid energy equation

$$\frac{\partial}{\partial t}\int_{y_1}^{y_2} \tfrac{1}{2}(u_+^2+v_+^2)\,dy=\int_{y_1}^{y_2}(\overline{-u_+v_+})\frac{\partial \bar{U}(y,t)}{\partial y}\,dy \qquad (8)$$

Here, u_+ and v_+ are periodic disturbance speeds and $\overline{u_+v_+}$ denotes a cyclic average. We now assume a disturbance similar in shape to that of linear theory, but modified by a time-dependent amplitude $B(t)$. Note that the left side of Eq. (8) equals $\partial E/\partial t$, while the $\overline{-u_+v_+}$ equals the τ in Eq. (4). Substitution in Eq. (8) and integration by parts leads to

$$\frac{\partial E}{\partial t}=2\omega_i E\int_{y_1}^{y_2}\int_{y_1}^{y}\frac{U_l''\,|\varphi_N|^2}{|U_l-c|^2}dy\frac{\partial \bar{U}}{\partial y}dy$$

$$=2\omega_i E\left[\bar{U}\int_{y_1}^{y}\frac{U_l''\,|\varphi_N|^2}{|U_l-c|^2}dy\,\Big|_{y_1}^{y_2}-\int_{y_1}^{y_2}\bar{U}\frac{U_l''\,|\varphi_N|^2}{|U_l-c|^2}dy\right] \qquad (9)$$

The first bracketed term vanishes since $\tau(y_1)=\tau(y_2)=0$. Use of Eq. (6) in the remaining integral yields

$$\frac{\partial E}{\partial t}=-2\omega_i E\left[\int_{y_1}^{y}\frac{U_lU_l''\,|\varphi_N|^2}{|U_l-c|^2}dy+E\int_{y_1}^{y_2}\frac{U_l''^2\,|\varphi_N|^4}{|U_l-c|^4}dy\right] \qquad (10)$$

For self-excited waves the real and imaginary parts of the complex equation used to derive Rayleigh's theorem imply that

$$-\int_{y_1}^{y_2}\frac{U_lU_l''\,|\varphi_N|^2}{|U_l-c|^2}dy=\int_{y_1}^{y_2}\{\,|\varphi_N'|^2+k^2|\varphi_N|^2\}dy \qquad (11)$$

Since the right side has been normalized to unity, the wave energy equation becomes

$$\frac{\partial E}{\partial t}=2\omega_i E\left[1-E\int_{y_1}^{y_2}\frac{U_l''^2\,|\varphi_N|^4}{|U_l-c|^4}dy\right] \qquad (12)$$

Equation (12) gives the first Landau constant explicitly. Also, since the above integral is positive we conclude that, to this order, nonlinearity is stabilizing and that equilibrium solutions can be found. In fact, the equilibrium wave energy density satisfies

$$E=\frac{1}{\int_{y_1}^{y_2}\frac{U_l''^2\,|\varphi_N|^4}{|U_l-c|^4}dy} \qquad (13)$$

where, again, φ_N is a normalized eigenfunction based upon U_l and k, which are prescribed. The solution for $\bar{U}(y,t)$ is obtained by solving Eq. (12) for E and combining with Eq. (6). The equilibrium solution for $\bar{U}(y,t)$ is just

$$\bar{U}=U_l(y)+\frac{U_l''\,|\varphi_N|^2}{\left|U_l-c\right|^2\int_{y_1}^{y_2}\frac{U_l''^2\,|\varphi_N|^4}{|U_l-c|^4}dy} \qquad (14)$$

Discussion and Conclusion

This Note examines the nonlinear stability problem for an inviscid parallel flow and, in particular, the effect of wave-induced mean flow distortion on the overall flow stability. Simple governing equations for the wave energy density and the mean velocity profile are derived which suggest that, to the order considered, the effect of this distortion is stabilizing. These equations are then solved, approximately, using the method described in Stuart,[3] and solutions for the equilibrium wave energy density and the equilibrium mean velocity profile are given in closed form.

It is important that we point out several limitations implicit in our approach. First, the neglect of higher order harmonics was made for mathematical expediency only, so that analytical solutions are possible; thus, the method describes neither the harmonic generation expected in nonlinear processes nor the distortion of the primary wave as would be induced by the neglected harmonics. These additional effects can, in some cases, significantly modify the Landau constant obtained here. Second, we stress that our analysis is a weakly nonlinear one only, restricted to waves close to the neutral curve; this limitation arises because we have, in our solution method, assumed a "wave shape" similar to that of linear theory. Also, because the solution obtained here is expanded about linear theory, the well-known restriction to growing waves with $c_i>0$ applies; solution to Rayleigh's equation, when viewed within the framework of the more complete Orr-Sommerfeld equation, are physically meaningful only for self-excited waves.[1] Finally, we stress that our analysis tacitly assumes normal modes. For linear flows, it is known that the flow associated with the continuous spectrum decays with time,[5] thereby justifying its neglect; this may or may not be the case in the weakly nonlinear problem considered here, and further study is needed.

The study of inviscid flow stability theory for high Reynolds number flows is, in itself, a well-accepted fluid-mechanical discipline; also, some recent work of Landahl[6] suggests that the breakdown mechanism leading to turbulent transition may well be inviscid. In this Note, however, the study was motivated by the possibility that equilibrium solutions may exist inviscidly, without the need to assume slow temporal variations as in Stuart's[3] original viscous approach. This possibility is demonstrated here where, as in Ref. 3, higher order harmonics are neglected. In closing we point out that the mean flow Ansatz supplied in Eq. (6) may be used in a Whitham-type solution[7] for slowly varying wave trains of wave number $k(x,t)$ and energy density $E(x,t)$ propagating in space and time. In this approach, an equation for wave crest conservation using the real part of the complex frequency only, but with a nonlinear Stokes' correction for the real frequency, is coupled with Eq. (12) but modified by a spatial flux term. This "straightforward" generalization of Whitham's conservative wave formalism is rigorously justified in a recently published paper for general kinematic waves,[8] but detailed calculations have not yet been pursued.

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Jet Thickness in High-Speed Two-Dimensional Planing

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Nomenclature

- C_R $= R/\frac{1}{2}\rho u_0^2 l$
- D_j = jet resistance
- D_w = wave resistance per unit width
- F_l = Froude number $= u_0/\sqrt{gl}$
- g = acceleration due to gravity
- l = wetted length
- L = lift force per unit width $= R\cos\tau$ for a flat plate
- R = resultant force per unit width
- u_0 = freestream velocity
- δ = jet thickness
- ρ = mass density of the fluid
- τ = trim angle
- β = angle of the resultant force R aft of a normal to the undisturbed flow ($\beta=\tau$ for a flat plate)
- θ = angle of the jet above the horizontal for a nonplanar plate ($\theta=\tau$ for a flat plate)

PAYNE[3] has shown that the high-speed wave drag of any two-dimensional pressure field is, per unit width

$$D_w=(L^2/l^2)\cdot(1/\rho g)\cdot(1/F_l^4) \tag{1}$$

The jet drag[4] is, in the two-dimensional case

$$D_J=\rho u_0^2\delta(1+\cos\tau) \tag{2}$$

Now for a flat plate in inviscid flow

$$D_w+D_J=R\sin\tau \tag{3}$$

Substituting for D_w and D_j from Eqs. (1) and (2), and writing $L=R\cos\tau$

$$\rho u_0^2\delta(1+\cos\tau)=R\sin\tau-\frac{R^2\cos^2\tau}{l\rho g F_l^4}$$

$$\therefore\ \frac{\delta}{l}=\frac{C_R}{2(1+\cos\tau)}\left[\sin\tau-\frac{C_R\cos^2\tau}{2F_l^2}\right] \tag{4}$$

In the limit $F_l\to\infty$

$$(\delta/l)\to[C_R\sin\tau/2(1+\cos\tau)]=(C_R/2)\tan(\tau/2)$$

which is a form of Pierson and Leshnover's[2] Eq. (3), derived from both momentum theory and their exact solution of the flowfield. The additional terms in Eq. (4) (which reduce the jet thickness at finite Froude numbers) are due to the heavy fluid effect of wave drag.

Squire's[1] high-speed theory result is[5]

$$\frac{\delta}{l}\simeq\frac{\pi}{4}\tau^2\Big/\left[1+\frac{5}{8}\frac{\pi}{F_l^2}\right]^2 \tag{5}$$

based upon

$$C_R\simeq\pi\tau\Big/\left[1+\frac{5}{8}\frac{\pi}{F_l^2}\right] \tag{6}$$

If we substitute Eq. (6) for C_R into Eq. (4) we obtain Eq. (5), but multiplied by the factor

$$[1+(\pi/8F_l^2)]$$

Thus, for both theories

$$\frac{\delta}{l}\to\frac{\pi\tau^2}{2(1+\cos\tau)}\quad\text{as}\quad F_l\to\infty \tag{7}$$

which expresses the fact that, in the limit, all the resistance is due to jet drag.

For this limit, Eq. (7) is compared in Fig. 1 with the Schwarz-Christoffel transformation solution of Pierson and Leshnover,[2] who obtained

$$\frac{\delta}{l}=\pi\left[\frac{1+\cos\tau}{1-\cos\tau}-\log\left(\frac{1-\cos\tau}{2\cos\tau}\right)+\frac{\pi\sin\tau}{1-\cos\tau}\right]^{-1} \tag{8}$$

The agreement between Eqs. (7) and (8) is good for small trim angles, but the theories diverge above about $\tau=5$ deg.

This is because Pierson and Leshnover obtain a normal force coefficient for a plate planing on a weightless fluid

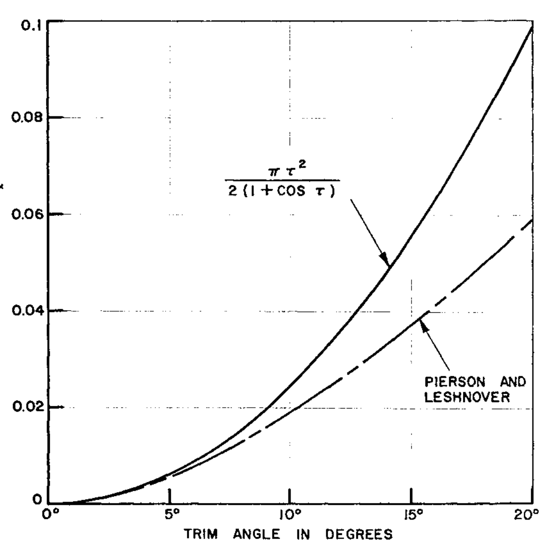

Fig. 1 Jet thickness as function of trim angle for limit of infinite Froude number. Pierson-Leshnover theory is for two-dimensional planing on weightless fluid.

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